Island size distribution in stochastic simulations of the Saffman-Taylor instability

1986 J. Phys. A: Math. Gen. 19 L195

(http://iopscience.iop.org/0305-4470/19/4/005)

J. Phys. A: Math. Gen. 19 (1986) L195-L200. Printed in Great Britain

LETTER TO THE EDITOR

Island size distribution in stochastic simulations of the Saffman–Taylor instability

J D Sherwood

Etudes et Fabrication Dowell Schlumberger, BP 90, 42003 St-Etienne Cedex 1, France

Received 14 November 1985

Abstract. A stochastic numerical scheme is used to simulate radial motion of a fluid interface through a two-dimensional porous medium. The point at which the interface moves is chosen with a probability proportional to the local pressure gradient. The interface can intersect itself, enclosing islands of the original fluid. The number of islands of area s is found to scale approximately as $s^{-\alpha}$, where α lies in the range 2.07 ± 0.7.

The interface between two fluids flowing in a porous medium is unstable when the more viscous fluid is displaced by a fluid of lower viscosity. Moreover, the unstable interface is continually excited by the randomly sized pores through which the fluid flows. This instability causes problems in oil reservoirs, into which water is injected to flush out the (more viscous) oil. It is therefore of commercial as well as scientific interest.

We shall first briefly describe a model (King and Scher 1985, DeGregoria 1985, Sherwood and Nittmann 1986) which has been developed in an attempt to simulate the motion of the interface. Although the model is closely connected with Darcy's law, its links with the physics of flow through individual pores of the rock are still unclear. However, here we shall be concerned with the model itself and, in particular, with the creation of islands of trapped fluid. If the advancing interface intersects itself, it completely surrounds an 'island', which is then preserved. Later we present results which suggest that the number of islands of area s scales as $s^{-2.07\pm0.07}$.

A continuum analysis of fluid flow in a porous medium is usually based on Darcy's law

$$u = -k\nabla p/\mu$$

where u is the fluid velocity, μ the fluid viscosity, p the pressure and k the permeability of the medium. If the fluid is incompressible, then $\nabla^2 p = 0$. We assume that two fluids are present, and that, at each point, the rock either contains fluid 1 or fluid 2. The interface between the fluids is considered to lie halfway between adjacent grid points containing the two different fluids. At the interface, the pressure and the normal component of velocity are continuous. (Note that the absence of a jump in pressure implies that no attempt has been made to introduce interfacial tension between the fluids.) A classical numerical scheme would first solve for the pressure. ∇p gives us the velocity everywhere and, in particular, the velocity of the interface. At each time step the interface would be advanced, along its entire length, by a distance proportional to the pressure gradient. In the discrete scheme presented here, the pressure field is

0305-4470/86/040195+06$02.50

first obtained by a standard finite difference method. The interface is then advanced at a *single* point chosen with probability proportional to the pressure gradient. The expected advance, at each point, is therefore proportional to that given by the continuum analysis. A similar model has been presented by Niemeyer *et al* (1984) to model dielectric breakdown. Their interface was held at constant potential, and their results therefore correspond to injection of fluid with zero viscosity.

We assume that the rock is initially saturated by fluid 1, with viscosity μ_1, and that fluid 2 is injected at a single point. The injection point is therefore at the centre of a region of rock occupied by fluid 2. At each time step this region grows, as the interface advances, and fluid 2 displaces fluid 1. Whenever a zone containing fluid 1 is surrounded by fluid 2, this zone is frozen and no further motion of its perimeter is allowed. Physically, it might be more realistic to allow this island of fluid 1 to be dragged along with the motion of the surrounding fluid (see Payatakes (1982) for a review of the motion of individual oil ganglia). However, the numerical scheme does not (at present) permit fluid 2 to be displaced by fluid 1, and such motion is not allowed.

The simulations were performed on a square grid, of size 361×361, though only points within a circle of radius 180 were considered. The central injection point was held at constant potential, as was the outer circle of grid points. At each time step, the pressure field p was first solved. The interface was then advanced, and fluid 2 replaced fluid 1 at one grid point. The Laplace equation for the pressure was solved by Gauss-Seidel iteration, since a good estimate of the solution was available from the previous time step. Relaxation was first performed over a grid of size 9×9 surrounding the point at which the interface had just moved. The entire grid was then relaxed, and it was this which took most of the computational time. At the high viscosity ratios ($\mu_1/\mu_2 \geqslant 100$) an additional intermediate grid of size 17×17 was also used. Simulations were stopped after 8000 time steps, or sooner, if fingers of injected fluid reached out halfway between the central point and the outer boundary. When the viscosity ratio was low, there was usually only one sweep of the entire grid at each time step. Even so, the simulations took 8–12 h on a VAX 11/785, of which all but approximately 25 min was taken up in Gauss-Seidel iteration.

Figure 1 shows typical examples of the region occupied by fluid 2. When $\mu_1 < \mu_2$ the interface is stable and few islands are formed. When $\mu_1 > \mu_2$ the interface is unstable and the growing fingers can surround much larger islands. We study the number of islands of area s (in grid block units) for a range of values of the viscosity ratio μ_1/μ_2. Since the interface between the fluids lies halfway between grid points, the area of each island is numerically the same as the number of grid points within it. The results presented are in all cases an average over five simulations. We shall see that it is meaningful to speak of the number of islands $n(s)$ created per 1000 time steps. For evaluation, we group the islands into size ranges 1, 2–3, 4–7, ..., and consider the sum

$$m(p) = \sum_{2^{p-1}}^{2^p - 1} n(s).$$

In the stable case, $\mu_1 = 0.1\mu_2$, the largest islands obtained were of area $s = 3$. Figure 2(a) shows the mean values of $m(p)$ after 1000, 2000, ..., 8000 time steps. The results for 2000–8000 steps indicate that a limit has been reached.

When $\mu_1 = \mu_2$ the effect of the square computational grid is evident (figure 1(b)), and there are no islands along the axes. The results for $m(p)$, shown on figure 2(b),

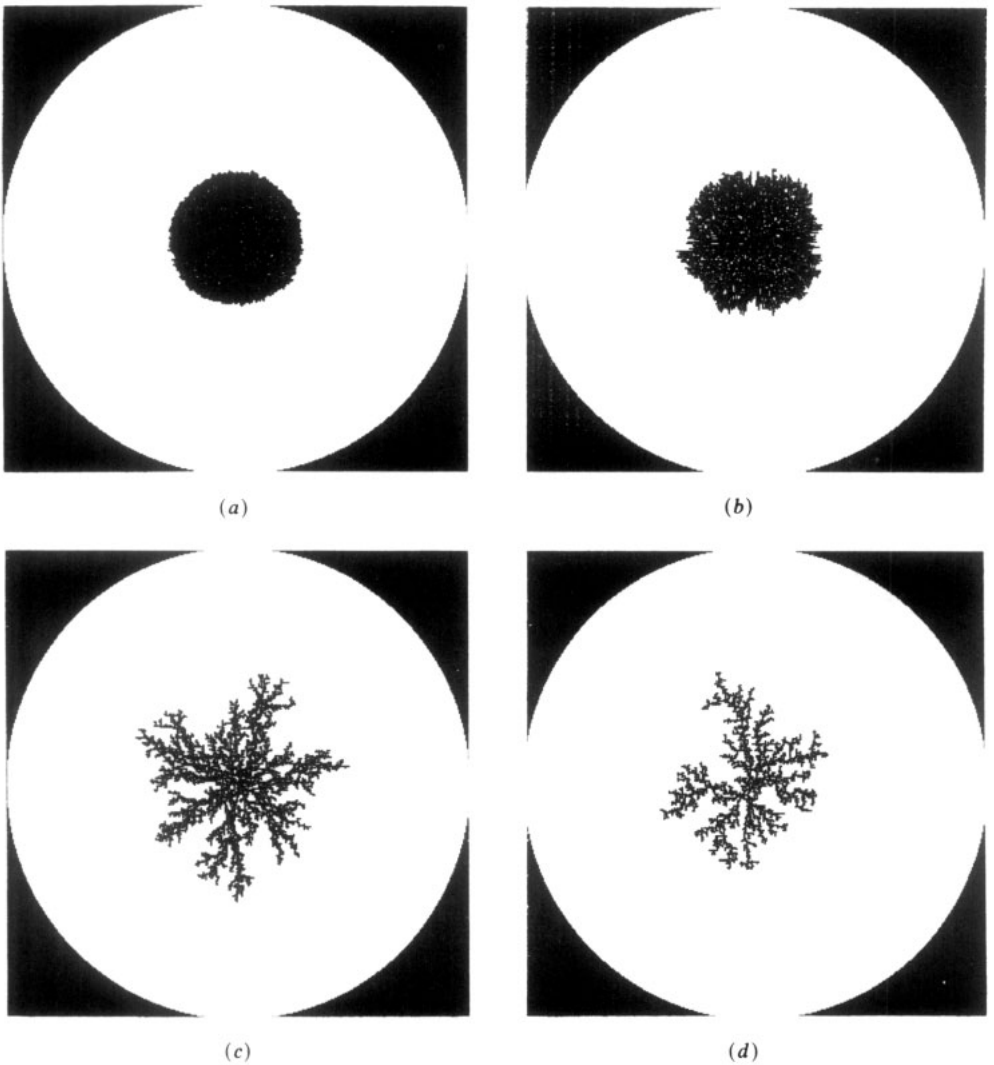

Figure 1. The region invaded by fluid 2, at viscosity ratios $\mu_1/\mu_2 = (a)$ 0.1, (b) 1.0, (c) 10^2, (d) 10^4. The number of time steps is (a) 8000, (b) 8000, (c) 6217, (d) 3424.

have not yet converged. At $\mu_1 = 2\mu_2$ this effect disappears, and $m(p)$ again converges satisfactorily (figure 2(c)).

In figure 3 we plot $\log_2(m(p))$ for viscosity ratios $\mu_1/\mu_2 = 10, 10^2, 10^3, 10^4$, and for times up to 8000 steps, 4000 steps, 2000 steps and 2000 steps respectively—at high viscosity ratios the fingers approach the outer boundary more rapidly and the simulation therefore stops sooner. For small p, a limit is rapidly approached as the number of time steps increases. For higher p the convergence is slow since it is impossible to have large islands at early times in the simulation. In all cases the results seem to approach limits which, to the limited accuracy available, seem to be straight lines of slope -1 (-1.07 ± 0.07 by eye). If we seek a distribution

$$n(s) \propto s^{-\alpha}$$

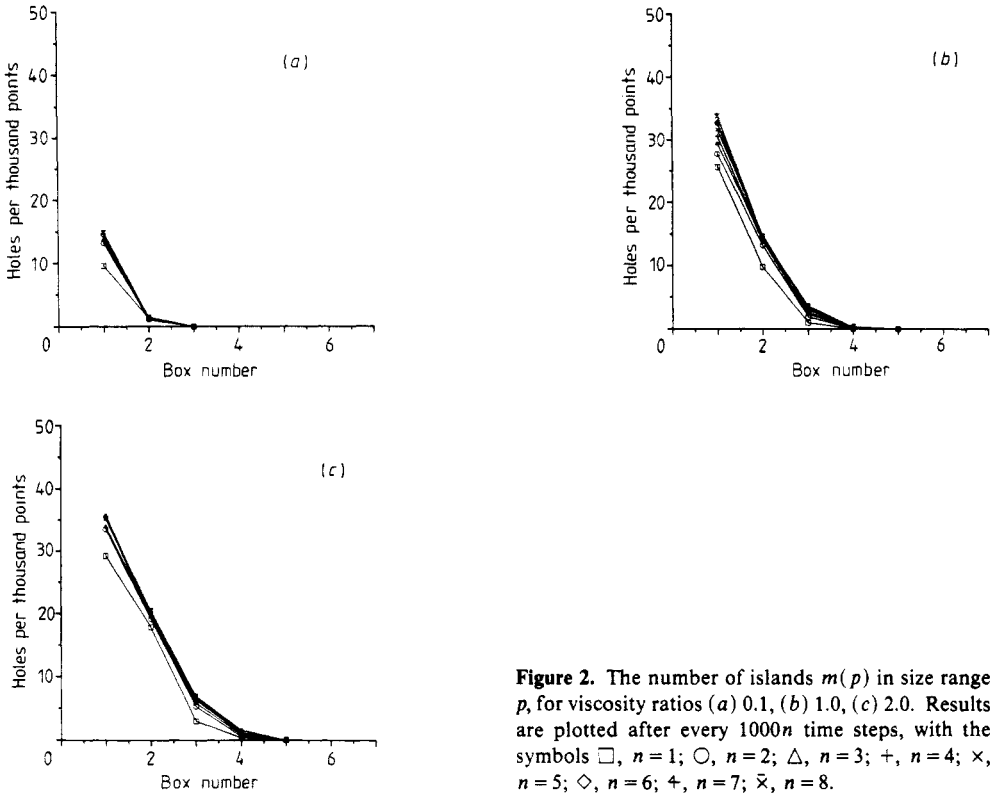

Figure 2. The number of islands $m(p)$ in size range p, for viscosity ratios (*a*) 0.1, (*b*) 1.0, (*c*) 2.0. Results are plotted after every $1000n$ time steps, with the symbols □, $n=1$; ○, $n=2$; △, $n=3$; +, $n=4$; ×, $n=5$; ◇, $n=6$; 4, $n=7$; x̄, $n=8$.

then, replacing sums by integrals we obtain the approximate result

$$m(p) \sim \frac{2^{(p-1)(1-\alpha)}}{1-\alpha}(2^{1-\alpha}-1).$$

Thus a straight line of slope -1 corresponds to $\alpha = 2$, while a slope -1.07 corresponds to $\alpha = 2.07$. Given the limited accuracy of the results, an attempt to express $n(s)$ in a more complicated form does not seem worthwhile. For the same reason, we cannot compare α to the exponent $\tau = 2.055$ in the cluster-size distribution at the two-dimensional percolation threshold (Stauffer 1985). To improve the statistics, a larger number of islands would be required, particularly for the large island sizes. This could only be obtained by increasing the number of time steps (which would in turn require that the outer boundary be further away).

We can consider some of the consequences of this distribution. Let the number of islands of area s and n time steps be $\beta n s^{-2}$. No islands are smaller than $s = 1$, and we assume that the largest island will be of area $O(\sqrt{n})$. Then the total area of the islands is

$$n\beta \sum_{s=1}^{\sqrt{n}} s^{-1} \sim \tfrac{1}{2} n\beta \log n.$$

The density of fluid 2 in the entire agglomerate of (fluid 2 + islands of fluid 1) is therefore

$$(1+\tfrac{1}{2}\beta \log n)^{-1}$$

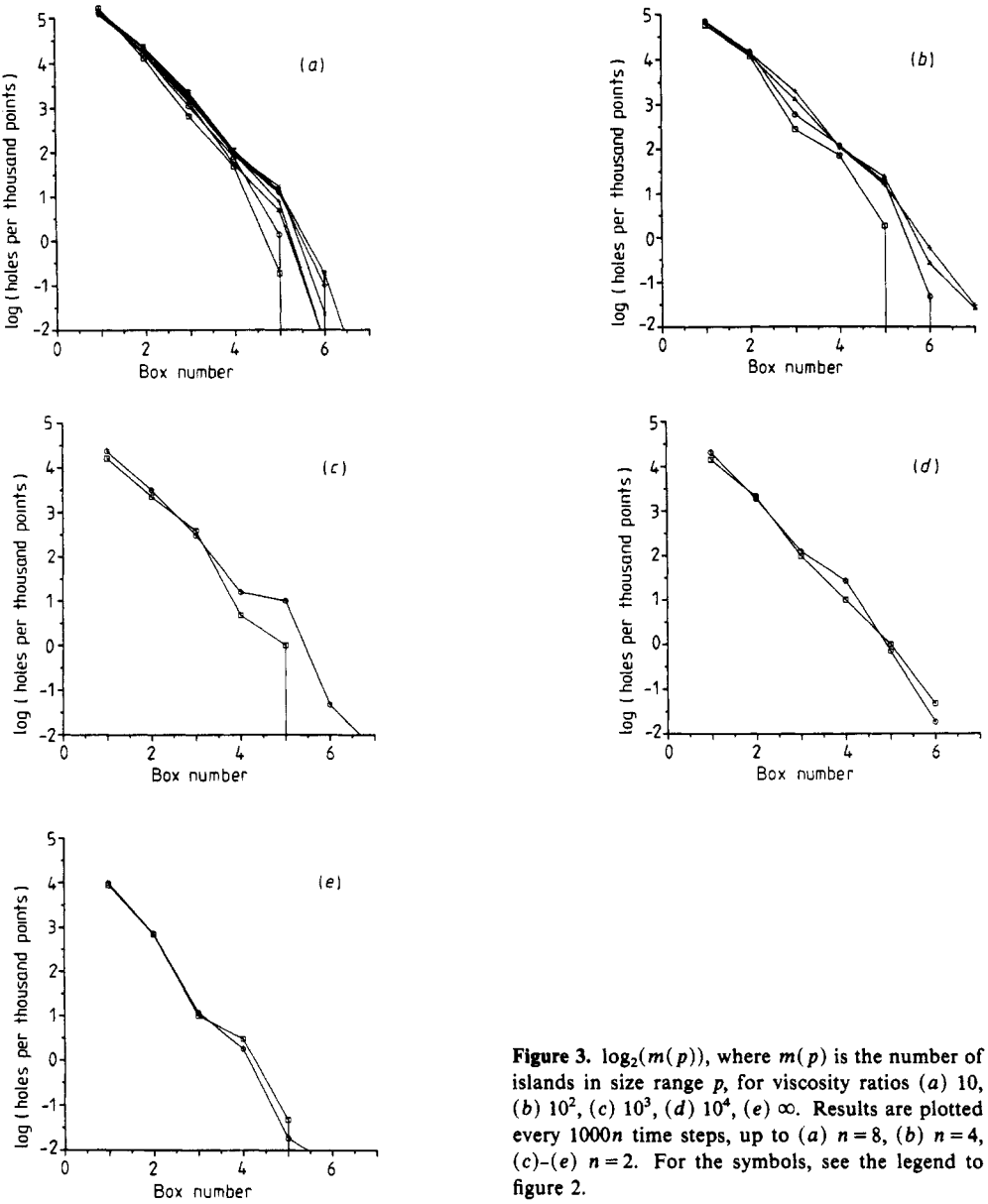

Figure 3. $\log_2(m(p))$, where $m(p)$ is the number of islands in size range p, for viscosity ratios (*a*) 10, (*b*) 10^2, (*c*) 10^3, (*d*) 10^4, (*e*) ∞. Results are plotted every $1000n$ time steps, up to (*a*) $n=8$, (*b*) $n=4$, (*c*)-(*e*) $n=2$. For the symbols, see the legend to figure 2.

which decreases only slowly with n. If we assume an island size distribution $\beta n s^{-2.07}$, the density again decreases slowly, as

$$\left(1+\beta\frac{(1-n^{-0.035})}{0.07}\right)^{-1}.$$

Note, however, that this sort of argument neglects entirely the original fluid 1 which remains within the fiords between the fingers of fluid 2.

If we subdivide all our grid lengths by a factor 2, we will increase n by a factor slightly less than 4 to cover the same physical area. The number of islands with areas

in the range $[4s, 4s+3]$ on the new fine scale will be approximately the same as the number of islands of area s on the old coarser scale. The main difference between simulations on the coarse and fine scales will be the appearance of small islands of area $s = 1$, 2 and 3 on the finer scale.

Simulations at an infinite viscosity ratio should be equivalent to simulations of diffusion-limited aggregation (Witten and Sander 1981, Paterson 1984, Kadanoff 1985). However, DLA rarely forms closed loops, since the diffusing particles only have a small probability of travelling down the fiords between fingers. This is also true in the simulations presented here when the injected fluid is inviscid, and therefore at uniform pressure (figure 3(e)). When the viscosity ratio is large, but finite, the simulations have the same general appearance as DLA (for the grid sizes considered here), but the number of closed loops is higher. In this case viscosity causes a pressure gradient along the length of each finger. The pressure gradient in the shorter slower fingers is lower, and there is a pressure difference between adjacent short and long fingers. Although the probability of the short finger growing towards the longer finger is small (at high viscosity ratios), islands can indeed be created when the number of time steps is large.

References

DeGregoria A J 1985 *Phys. Fluids* **28** 2933
Kadanoff L 1985 *J. Stat. Phys.* **39** 267
King M J and Scher H 1985 *SPE paper* 14366
Niemeyer L, Pietronero L and Wiesmann H J 1984 *Phys. Rev. Lett.* **52** 1033
Paterson L 1984 *Phys. Rev. Lett.* **52** 1621
Payatakes A C 1982 *Ann. Rev. Fluid Mech.* **14** 365
Sherwood J D and Nittmann J 1986 *J. Physique* in press
Stauffer D 1985 *Introduction to percolation theory* (London: Taylor and Francis)
Witten T A and Sander L M 1981 *Phys. Rev. Lett.* **47** 1400